Grothendieck-Riemann-Roch theorem, superconnections, and index theorem

joint work with Jean-Michel Bismut and Shu Shen

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Some notations

- Let X be a compact complex manifold without boundary, not necessarily Kähler.
- Let $\mathsf{K}_0(X)$ be the Grothendieck group of coherent sheaves on X.
- Let
$$\mathsf{H}_{\mathsf{BC}}^{(=)}(X,\mathbb{R})=\bigoplus_{p=0}^{\dim_{\mathbb{C}}X}\mathsf{H}_{\mathsf{BC}}^{p,p}(X,\mathbb{R})$$
be the real Bott-Chern cohomology of X (defined later).

Some constructions

- We can define the Chern character map via Chern-Weil theory
$$\mathrm{ch}_{\mathrm{BC}} : \mathrm{K}_0(X) \to \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R}).$$
- For a holomorphic map $f : X \to Y$ between compact complex manifolds, we can define the direct image map
$$f_! : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$$
and the push-forward map
$$f_* : \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R}) \to \mathrm{H}_{\mathrm{BC}}^{(=)}(Y, \mathbb{R}).$$

The main theorem

Theorem (Bismut-Shu-W, [Bismut et al., 2023])

For $\mathcal{F} \in K_0(X)$ *we have in* $H_{BC}^{(=)}(Y, \mathbb{R})$

$$Td_{BC}(TY)ch_{BC}(f_!\mathcal{F}) = f_*[Td_{BC}(TX)ch_{BC}(\mathcal{F})], \tag{1}$$

i.e. the following diagram commutes

$$\begin{tikzcd} K_0(X) \arrow[r, "f_!"] \arrow[d, "Td_{BC}(TX)ch_{BC}(-)"'] & K_0(Y) \arrow[d, "Td_{BC}(TY)ch_{BC}(-)"] \\ H_{BC}^{(=)}(X, \mathbb{R}) \arrow[r, "f_*"] & H_{BC}^{(=)}(Y, \mathbb{R}) \end{tikzcd} \tag{2}$$

- $\mathrm{Td}_{\mathrm{BC}}(TX) \in \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R})$ is the Todd class of the holomorphic tangent bundle.

Some comments

- The proof uses the heat kernel method.
- The proof of the most general case uses the heat kernel of a hypoelliptic Dirac operator.
- If X and Y are projective complex manifolds, then the result reduces to the classic Grothendieck-Riemann-Roch theorem.
- If Y is a point, then the result reduces to the Hirzebruch-Riemann-Roch theorem.

Contents

A quick review of the Chern-Weil theory of vector bundles

- Let X be a C^∞-manifold without boundary, let E be a C^∞-complex vector bundle on X.
- Let $\nabla_E : C^\infty(X, E) \to \Omega^1(X, E)$ be a smooth connection on E, then $(\nabla_E)^2 \in \Omega^2(X, \mathrm{End}(E))$.
- Chern-Weil: $\mathrm{ch}(E) := \mathrm{tr}[\exp(-(\nabla_E)^2/2\pi i)] \in \mathrm{H}^{\mathrm{even}}_{\mathrm{dR}}(X)$ is independent of the choice of ∇_E.
- If E has a metric and ∇_E is unitary with respect this metric, then $\mathrm{tr}[\exp(-(\nabla_E)^2/2\pi i)]$ is a real form.

Holomorphic vector bundles

Theorem (Koszul-Malgrange theorem)

Holomorphic vector bundles over a complex manifold X are equivalently complex C^∞-vector bundles E which are equipped with a $\bar\partial$-connection

$$\nabla^{E\prime\prime} : C^\infty(X,E) \to \Omega^{0,1}(X,E).$$

such that $(\nabla^{E\prime\prime})^2 = 0$.

- If E is equipped with a Hermitian metric h^E, then we can extend $\nabla^{E\prime\prime}$ to the unique Chern connection $\nabla^E = \nabla^{E\prime} + \nabla^{E\prime\prime}$ where $\nabla^{E\prime} : C^\infty(X,E) \to \Omega^{1,0}(X,E)$ and it is compatible with h^E in the sense that $\mathrm{d}h^E(s_1,s_2) = h^E(\nabla^E s_1, s_2) + h^E(s_1, \nabla^E s_2)$.
- For a Chern connection ∇^E we have $(\nabla^E)^2 \in \Omega^{1,1}(X, \mathrm{End}(E))$.
- $\mathrm{ch}(E,h^E) := \mathrm{tr}[\exp(-(\nabla_E)^2/2\pi i)]$ is de Rham closed and real.

The Chern character of holomorphic vector bundles

Theorem ([Bott and Chern, 1965])

For two different Hermitian metrics h^E and $\tilde{h}_E$, there exists a form $\gamma(E, h^E, \tilde{h}_E)$ such that

$$ch(E, h^E) - ch(E, \tilde{h}_E) = \bar{\partial}\partial\gamma(E, h^E, \tilde{h}_E).$$

- The Bott-Chern cohomology $\mathrm{H}_{\mathrm{BC}}^{p,q}(X, \mathbb{C})$ is defined to be
$$\mathrm{H}_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) := (\Omega^{p,q}(X, \mathbb{C}) \cap \ker d)/\bar{\partial}\partial\Omega^{p-1,q-1}(X, \mathbb{C}).$$
- We also have $\mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{C}) := \bigoplus_{p=0}^{\dim_{\mathbb{C}} X} \mathrm{H}_{\mathrm{BC}}^{p,p}(X, \mathbb{C})$. The space $\mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{C})$ has conjugations so we get the subspace $\mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R})$.
- Bott-Chern theorem shows that $\mathrm{ch}(E, h^E) \in \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{C})$ is independent of the metric h^E.

Discussion on Bott-Chern cohomology

- There is a canonical map $\mathrm{H}_{\mathrm{BC}}^{p,q}(X,\mathbb{C}) \to \mathrm{H}_{\mathrm{dR}}^{p+q}(X,\mathbb{C})$.
- If X is Kähler, then $\mathrm{H}_{\mathrm{BC}}(X,\mathbb{C}) \cong \mathrm{H}_{\mathrm{dR}}(X,\mathbb{C})$. (The $\partial\bar{\partial}$-lemma in complex geometry.)
- In general, $\mathrm{H}_{\mathrm{BC}}(X,\mathbb{C}) \neq \mathrm{H}_{\mathrm{dR}}(X,\mathbb{C})$ (Example: Iwasawa manifold, [Nakamura, 1975] and [Angella, 2013]).
- H_{BC} is clearly functorial with respect to pullback.
- We can also define H_{BC} by *currents* instead of forms (elliptic regularity theory).
- Then for a proper holomorphic maps $f : X \to Y$, we can define the pushforward $f_* : \mathrm{H}_{\mathrm{BC}}(X,\mathbb{C}) \to \mathrm{H}_{\mathrm{BC}}(Y,\mathbb{C})$. The push forward is also functorial.

Complexes of holomorphic vector bundles

In the view of the Koszul-Malgrange theorem, a cochain complex of finite dimensional holomorphic vector bundles corresponds to the following data:

- a $\mathbb{Z}$-graded C^∞-vector bundle $E^\bullet$;
- a C^∞-map $v : E^i \to E^{i+1}$ and a map $\nabla^{E^i \prime\prime} : C^\infty(X, E^i) \to \Omega^{0,1}(X, E^i)$ such that

  $$v^2 = 0, (\nabla^{E^i \prime\prime})^2 = 0 \text{ and } [v, \nabla^{E^i \prime\prime}] = 0.$$

  In other words $(v + \nabla^{E^i \prime\prime})^2 = 0$.
- We modified the ± sign for later applications.
- $A'' := v + \nabla^{E^i \prime\prime}$ is an example of *antiholomorphic flat superconnections* which will be defined later.

The Chern connection of complexes of holomorphic vector bundles

- Let h^E be a $\mathbb{Z}$-graded Hermitian metric.
- $v^* : E^i \to E^{i-1}$ the adjoint of v under h^E.
- $\nabla^{E^i\prime} : C^\infty(X, E) \to \Omega^{1,0}(X, E^i)$ such that $\nabla^{E^i\prime} + \nabla^{E^i\prime\prime}$ is a Chern connection on E^i.
- Let $A' = v^* + \nabla^{E^i\prime}$ and $A = A' + A''$(example of superconnection). We have
$$(A')^2 = 0,\ (A'')^2 = 0, \text{ hence } A^2 = [A', A''].$$
- Warning: A^2 is not necessarily in $\Omega^{1,1}$.

The Chern character of complexes of holomorphic vector bundles

- As before $A = A' + A'' = v + v^* + \nabla^{E^i\prime} + \nabla^{E^i\prime\prime}$.
- ch$(E, h^E) := \frac{1}{(2\pi i)^N}tr_s \exp(-A^2)$. Here N is the number operator and tr$_s$ is the supertrace.

Theorem (Bismut-Gillet-Soulé 1988, [Bismut et al., 1988])

1. *ch*$(E, h^E) \in \Omega^{(=)}(X, \mathbb{R})$ *and is d-closed.*
2. *ch*$(E, h^E) \in H_{BC}^{(=)}(X, \mathbb{R})$ *is independent of the metric* h^E.
3. *ch*$(E, h^E) = \sum_i (-1)^i ch(E^i, h^{E^i}) \in H_{BC}^{(=)}(X, \mathbb{R})$.

- As a consequence, if the cochain complex $(E^\bullet, v)$ is exact, then ch$(E, h^E) = 0 \in \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R})$. (Consider $h_t^{E^i} := t^i h^{E^i}$, A'' remains unchanged and $A' = tv^* + \nabla^{E^i\prime}$. After conjugation, $A' + A''$ becomes $\sqrt{t}(v + v^*) + \nabla^{E^i\prime} + \nabla^{E^i\prime\prime}$. Then compute ch$(E, h_t^E)$ and let $t \to +\infty$.)

Coherent sheaves

- The category if holomorphic vector bundles is not good in the sense that it does not admit kernels and cokernels, i.e. it is an exact category but not an abelian category.

Definition

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called a **coherent sheaf** if it satisfies the following two conditions.

- $\mathcal{F}$ is of finite type over $\mathcal{O}_X$, that is, every point in X has an open neighborhood U in X such that there is a surjective morphism $\mathcal{O}_X^n|_U \twoheadrightarrow \mathcal{F}|_U$ for some $n > 0$;
- for *any* open set $U \subseteq X$, *any* $n > 0$, and *any* morphism $\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.

More on coherent sheaves

- The category of coherent sheaves is abelian.
- Oka's coherence principle: Let $(X, \mathcal{O}_X)$ be a complex manifold with holomorphic functions. Then $\mathcal{O}_X$ itself is coherent. $\Rightarrow$ Any finitely generated locally free sheaf (i.e. finite dimensional holomorphic vector bundle) is coherent.
- Not all coherent sheaves are locally free, for example, skyscraper sheaves, ideal sheaves, etc.

Bounded derived category of coherent sheaves

- $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ the bounded derived category of coherent sheaves on X.
- Objects: bounded cochain complexes of $\mathcal{O}_X$-modules with coherent cohomologies.
- Morphisms: has the universal property that quasi-isomorphisms are invertible in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.
- $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a triangulated category: it has the shift-by-1 and exact triangles.
- On Grothendieck groups we have $\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) \cong K_0(X)$.
- For a proper holomorphic map $X \to Y$, we can define the derived pushforwad functor $\mathrm{R}f_* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ via soft resolutions.
- This induces a group homomorphism $f_! : K_0(X) \to K_0(Y)$.

Coherent sheaves and holomorphic vector bundles

- Syzygy theorem: Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for a complex manifold X. Every point in X has an open neighborhood U in X such that there is a locally free resolution $E^{\bullet} \overset{\sim}{\to} \mathcal{F}|_U$ where $E^{\bullet}$ is a bounded complex of holomorphic vector bundles on U.
- It does not hold if X is singular.
- The locally free resolution does not always exist *globally* on X if X is not projective, for counter examples see [Voisin, 2002].
- Let $G_0(X)$ be the Grothendieck group of holomorphic vector bundles on X. In general $\mathrm{K}_0(X) \ncong G_0(X)$.
- By Bismut-Gillet-Soulé, $\mathrm{ch}(-)$ gives a map $G_0(X) \to \mathrm{H}^{(=)}_{\mathrm{BC}}(X, \mathbb{R})$.
- In this talk we focus on $\mathrm{K}_0(X)$. Is there a Chern character map $\mathrm{ch} : K_0(X) \to \mathrm{H}^{(=)}_{\mathrm{BC}}(X, \mathbb{R})$?

Antiholomorphic superconnection

Definition (Antiholomorphic superconnection)

Let X be a complex manifold and $E^\bullet$ be a $\mathbb{Z}$-graded bounded C^∞-vector bundles on X. A superconnection A'' is a differential operator of total degree 1 acting on $\Omega^{\bullet,\bullet}(X, E^\bullet)$ which satisfies the Leibniz rule

$$A''(\omega s) = \bar{\partial}(\omega) \cdot s + (-1)^{|\omega|}\omega \wedge A''(s).$$

- By the Leibniz rule A'' is determined by its restriction to $C^\infty(X, E^\bullet)$.
- $A'' = v_0 + \nabla^{E''} + v_2 + \dots$ where $v_i \in \Omega^{0,i}(X, \mathrm{End}^{1-i}(E^\bullet))$ and $\nabla^{E''}$ is an antiholomorphic connection on each E^i.
- $v_0 : C^\infty(X, E^i) \to C^\infty(X, E^{i+1})$, $v_2 : C^\infty(X, E^i) \to \Omega^{0,2}(X, E^{i-1})$, ...

Antiholomorphic superconnection illustration

This is an illustration:

$$\begin{tikzcd}
 & \dots & C^\infty(X,E^i) \arrow[r, "v_0"] \arrow[d, "\nabla^{E^i\prime\prime}"] \arrow[ddl, "v_2"'] \arrow[dddll, "v_3"'] & C^\infty(X,E^{i+1}) \arrow[r, "v_0"] \arrow[d, "\nabla^{E^{i+1}\prime\prime}"] & \dots \\
 & \dots & \Omega^{0,1}(X,E^i) \arrow[r, "v_0"] \arrow[d, "\nabla^{E^i\prime\prime}"] & \Omega^{0,1}(X,E^{i+1}) \arrow[r, "v_0"] \arrow[d, "\nabla^{E^{i+1}\prime\prime}"] & \dots \\
 & \Omega^{0,2}(X,E^{i-1}) \arrow[r, "v_0"] \arrow[d] & \Omega^{0,2}(X,E^i) \arrow[r, "v_0"] \arrow[d] & \Omega^{0,2}(X,E^{i+1}) \arrow[r] \arrow[d] & \dots \\
\dots & \dots & \dots & \dots &
\end{tikzcd}$$

Antiholomorphic flat superconnections

Definition ([Block, 2010])

An antiholomorphic flat superconnection on a complex manifold X is a $\mathbb{Z}$-graded bounded C^∞-vector bundles $E^\bullet$ together with a *flat* antiholomorphic superconnection A''. Flat means $(A'')^2 = 0$.

- Componentwisely, flatness means that
  0. $v_0^2 = 0$: $(E^\bullet, v_0)$ is a cochain complex of C^∞-vector bundles;
  1. $[\nabla^{E''}, v_0] = 0$: v_0 and $\nabla^{E''}$ are compatible;
  2. $(\nabla^{E''})^2 + [v_0, v_2] = 0$: $\nabla^{E''}$ is flat *up to cochain homotopy*.

  ...
- In this talk all antiholomorphic superconnections are flat.

Antiholomorphic flat superconnections and coherent sheaves

- Antiholomorphic flat superconnections on X form a dg-category $\mathsf{B}(X)$ which has a pre-triangulated structure.
- $\mathsf{B}(X)$ has the shift-by-1 functor and mapping cones.

Theorem ([Block, 2010])

For a compact complex manifold X, $\underline{\mathsf{B}}(X)$, the homotopy category of $\mathsf{B}(X)$, is triangulated equivalent to $D^b_{coh}(X)$. In other words, $\mathsf{B}(X)$ gives a dg-enhancement of $D^b_{coh}(X)$.

- In particular, a coherent sheaf corresponds to an antiholomorphic flat superconnection which is unique up to *homotopy equivalence*.
- Chuang, Holstein, and Lazarev ([Chuang et al., 2021]) generalized this result to non-compact complex manifolds with slightly modified $D^b_{coh}(X)$.

Idea of Block's construction

- For an object $(E^\bullet, A'')$ in $B(X)$, we define a cochain complex of sheaves $(\mathcal{F}^\bullet, d)$ with
$$\mathcal{F}^n(U) = \oplus_{p+q=n}\Gamma(U, \Omega^{0,p}(E^q))$$
and $d : \mathcal{F}^n \to \mathcal{F}^{n+1}$ is given by A''.
- $(\mathcal{F}^\bullet, d)$ is a cochain complex of sheaves of $\mathcal{O}_X$-modules.
- Each $\mathcal{F}^n$ is *not* coherent, but we can prove that its cohomology sheaves are all coherent.
- For each object $(\mathcal{F}^\bullet, d)$ in $D^b_{coh}(X)$, we can find (in a not quite constructive way) a $(E^\bullet, A'')$ corresponding to it.

A pairing on forms

- We equip $E^\bullet$ with a degreewise Hermitian metric h^E.
- We extend h^E to a pairing on $\Omega^\bullet(X, E^\bullet)$: For $\alpha = \omega \otimes e$ and $\beta = \eta \otimes f$, we define

$$h^E(\alpha, \beta) := h^E(e, f)\omega \wedge \bar{\eta}.$$

- For compact X and $\alpha,\ \beta \in \Omega^\bullet(X, E^\bullet)$, we define a pairing

$$\theta_{h^E}(\alpha, \beta) := (-1)^{|\alpha|(|\alpha|+1)}\left(\frac{i}{2\pi}\right)^{\dim_{\mathbb{C}} X}\int_X h^E(\alpha, \beta).$$

The Chern superconnection

Proposition ([Qiang, 2017]; [Bismut et al., 2023])

For an antiholomorphic flat superconnection A'', there exists a unique holomorphic flat superconnection A' which is the adjoint of A'' with respect to θ_{h^E}.

- It is the "super" version of the existence and uniqueness of the Chern connection.
- $A' = v_0^* + \nabla^{E\prime} + v_2^* + \ldots.$
- $v_0^* : C^\infty(X, E^i) \to C^\infty(X, E^{i-1})$, $v_2^* : C^\infty(X, E^i) \to \Omega^{2,0}(X, E^{i+1})$, $\ldots$
- It gives a unified construction of v_i^* and $\nabla^{E\prime}$.

The Chern character

Let $A := A' + A''$. Then $A^2 = [A', A'']$.

Definition (Chern character)

For an antiholomorphic flat superconnection $(E^\bullet, A'')$ with a metric h^E, we define the Chern character to be

$$\mathrm{ch}(E^\bullet, A'', h^E) := \left(\frac{1}{2\pi i}\right)^{N/2} \mathrm{tr}_s[\exp(-A^2)]. \tag{3}$$

Theorem ([Qiang, 2017]; [Bismut et al., 2023])

- *$ch(E^\bullet, A'', h^E) \in \Omega^{(=)}(X, \mathbb{R})$ and is d-closed;*
- *$ch(E^\bullet, A'', h^E) \in H_{BC}^{(=)}(X, \mathbb{R})$ is independent of the choice of h^E.*

The Chern character from $K_0(X)$

Proposition ([Qiang, 2017]; [Bismut et al., 2023])

If the cochain complex $(E^\bullet, v_0)$ *is exact, then* $ch(E^\bullet, A'', h^E) = 0 \in H_{BC}^{(=)}(X, \mathbb{R})$.

- This proposition+cone construction ⇒ we obtain a map

$$\mathrm{ch} : K_0(X) \to \mathrm{H}_{\mathrm{BC}}^{(=)}(X, \mathbb{R}).$$

Proposition ([Qiang, 2017]; [Bismut et al., 2023])

The map ch $: K_0(X) \to H_{BC}^{(=)}(X, \mathbb{R})$ *is a ring homomorphism and compatible with pullbacks.*

Other approaches

- Simplicial method: Toledo-Tong (1978), Green (1980), Hosgood (2020) (de Rham cohomology).
- Axiomatic method: Grivaux (2010) (rational Deligne cohomology), Wu (2020) (rational Bott-Chern cohomology).
- Grivaux unicity theorem: all constructions are compatible.

Main steps of the proof

- A holomorphic map $f : X \to Y$ can be factorized as
$$X \xrightarrow{i} X \times Y \xrightarrow{p} Y$$
where $i(x) = (x, f(x)) \in X \times Y$ is the graph embedding and π is the projection.
- $f_!$ on K_0 and f_* on $\mathrm{H}_{\mathrm{BC}}^{(=)}$ are both functorial.
- Hence it is sufficient to proof the Grothendieck-Riemann-Roch theorem in the following two cases
  1. Embeddings;
  2. Projections of the form $X \times Y \to Y$.
- A central problem in both cases is how to compute $f_! : K_0(X) \to K_0(Y)$ in the framework of antiholomorphic flat superconnections.

The embedding case (easy case)

Theorem (Bismut-Shen-W, [Bismut et al., 2023])

Let $i : X \hookrightarrow Y$ be an embedding and $N_{X/Y}$ be the normal bundle. Then we have

$$\mathit{ch}(i_! \mathcal{F}) = i_* \frac{\mathit{ch}(\mathcal{F})}{\mathit{Td}(N_{X/Y})} \text{ in } H_{BC}^{(=)}(Y).$$

- In the special case that $Y = \mathbb{P}(E)$ where E is a holomorphic vector bundle on X, and the embedding is

$$X \overset{\cong}{\to} X \times \{0\} \hookrightarrow \mathbb{P}(E);$$

1. We can compute $\mathrm{R}i_*$ of antiholomorphic flat superconnections via Koszul resolution;
2. We can compute the Todd class;
3. The theorem above is a direct check in this case.

The embedding case (easy case), continued

- For general embeddings, it is difficult to compute Ri_* of antiholomorphic flat superconnections.
- Nevertheless, we use *deformation to the normal cone* + the functoriality of K_0 and $\mathrm{H}_{\mathrm{BC}}^{(=)}$ to reduce to the previous case.
- Rough idea: We construct a pullback diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, hook, "\tilde{i}"] \arrow[d, two heads, "\pi_X"] & \tilde{Y} \arrow[d, two heads, "\pi_Y"] \\
X \arrow[r, hook, "i"] & Y
\end{tikzcd}$$

  such that $\tilde{Y} = \mathbb{P}(E)$ for a vector bundle E on $\tilde{X}$.
- We then use $\pi_Y^* \circ i_* = \tilde{i}_* \circ \pi_X^*$.
- It is a Grothendieck style proof.

A review of the family index theorem

- Let $M \to S$ be a family of manifolds modeled by X.
- Consider a $\mathbb{Z}/2$-graded C^∞-vector bundle E on M.
- Let D be a family of Dirac operator in the X direction on E.
- In good case ind(D) is a $\mathbb{Z}/2$-graded vector bundle on S. In any case $[\text{ind}(D)] \in K_0(S)$.

Theorem (Family index theorem, [Berline et al., 2004])

$ch(ind(D)) = \int_{M/S} \hat{A}(M/S)ch(E/\mathscr{S})$ in $H^\bullet(S)$.

- Go back to complex geometry, $\bar{\partial}_X + \bar{\partial}_X^*$ is a Dirac operator in the X direction, which shows the relation between Atiyah-Singer and Riemann-Roch.

The derived pushforward of antiholomorphic flat superconnections under projections

- Let $M = X \times S$ and consider $p : X \times S \to S$ and $q : X \times S \to X$ (We use S instead of Y from now on).
- Let $(E^\bullet, A'')$ be an antihomorphic flat superconnection on $X \times S$.
- The associated sheaf $\mathcal{F}^\bullet$ is soft.
- We take $\mathrm{R}p_*(E^\bullet, A'') = (\underline{p_*}(E^\bullet), \underline{p_*}(A''))$ where
  - $\underline{p_*}(E^\bullet)$ is a bounded $\mathbb{Z}$-graded infinite dimensional C^∞-vector bundle on S whose fiber at $s \in S$ is
  $$\Omega^{0,\bullet}(X \times \{s\}, E^\bullet);$$
  - $\underline{p_*}(A'')$ is the same as A'' but *regraded*.
- Example: $E = \mathbb{C}_{X \times S}$ and $A'' = \bar{\partial}^{X \times S}$ which is a connection. Then
  - $\underline{p_*}(E) = \Omega^{0,\bullet}(X)$, an infinite-dimensional trivial bundle on S;
  - $\underline{p_*}(A'') = \bar{\partial}^X + \bar{\partial}^S$ which is a superconnection.

The Chern connection of the derived pushforward

- Fix a metric h^E on $E^\bullet$ and a Riemannian metric g^X on X.
- They induce a metric on the infinite dimensional vector bundle $\underline{p_*}(E^\bullet)$ by integration over X.
- Let $\underline{p_*}(A')$ be the adjoint of $\underline{p_*}(A'')$ with respect to the above metric.
- Example: $E = \mathbb{C}_{X\times S}$ and $A'' = \bar{\partial}^{X\times S}$, then
    - $A' = \partial^{X\times S} = \partial^X + \partial^S$;
    - $\underline{p_*}(A') = (\bar{\partial}^X)^* + \partial^S$.

Some further discussion on $(\underline{p_*}(E^\bullet), \underline{p_*}(A''))$

- $[\underline{p_*}(A'), \underline{p_*}(A'')]$ is a second-order elliptic operator in the X direction.
- On each small neighborhood in S, we can use *spectral truncation* to truncate $(\underline{p_*}(E^\bullet), \underline{p_*}(A''))$ to a finite dimensional antiholomorphic flat superconnection.
- Consequence: $(\underline{p_*}(E^\bullet), \underline{p_*}(A''))$ gives a object in $D^b_{coh}(S)$.
- This gives an alternative proof of the *Grauert direct image theorem* [Grauert and Remmert, 1984] in our case.

The Chern character of the derived pushforward

- $\exp(-[\underline{p_*}(A'), \underline{p_*}(A'')])$ is a trace-class operator on the infinite dimensional vector bundle $\underline{p_*}(E^\bullet)$.
- We define
$$\mathrm{ch}(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E) = \left(\frac{1}{2\pi i}\right)^{N_S/2} \mathrm{tr_s}[\exp(-[\underline{p_*}(A'), \underline{p_*}(A'')])].$$
- $\mathrm{ch}(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E)$ is a differential form on S.

Properties of the Chern character

Theorem (Bismut-Shen-W, [Bismut et al., 2023])

- *ch*$(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E) \in \Omega^{(=)}(S, \mathbb{R})$ *and is d-closed;*
- *ch*$(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E) \in H_{BC}^{(=)}(S, \mathbb{R})$ *is independent of the choice of* h^E *and* g^X*;*
- *ch*$(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E) =$ *ch*$(Rp_*(E^\bullet, A'')) \in H_{BC}^{(=)}(S, \mathbb{R})$.

- The third point in the theorem means that for another (finite dimensional) representative of R$p_*(E^\bullet, A'')$in B(S), its Chern character is the same as ch$(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X, h^E)$.
- If R$p_*(E^\bullet, A'')$ has locally free cohomology sheaves which we denote by $\mathcal{H}$R$p_*(E^\bullet)$, then $\lim_{t\to+\infty}$ ch$(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X/t, h_t^E)$ exists and equals ch$(\mathcal{H}$R$p_*(E^\bullet, A''))$ as differential forms.

The Grothendiec-Riemann-Roch theorem for projections: special case

- We also have ch$(E^\bullet, A'', h^E) \in \Omega^{(=)}(X \times S, \mathbb{R})$.
- The Riemannian metric g^X on X induces a $(1,1)$-form ω_X on X which is not closed in general.

Theorem (Bismut-Shen-W, [Bismut et al., 2023])

If $\bar{\partial}\partial\omega_X = 0$, *then in* $\Omega^{(=)}(S, \mathbb{R})$ *we have*

$$\lim_{t \to 0+} ch(\underline{p_*}(E^\bullet), \underline{p_*}(A''), g^X/t, h^E) = \int_X Td(X) ch(E^\bullet, A'', h^E). \qquad (4)$$

- It includes the case that X is Kähler.

Idea of the proof

- Getzler's rescaling: we give the following degrees to operators:
  1. $\partial/\partial t$: degree 2;
  2. First order differential operators along X: degree 1;
  3. The Clifford element $c(e)$, $e \in TX$: degree 1;
- Lichnerowicz formula ⇒ If $\bar{\partial}\partial\omega_X = 0$, then $([\underline{p_*}(A', \underline{p_*}(A''))])^2$ is of degree ≤ 2.
- When $t \to 0+$, we can ignore the lower degree terms and then compute the limit.

The total space of tangent bundle

- If $\bar{\partial}\partial\omega_X \neq 0$, then we need to use *hypoelliptic superconnections*.
- Let $\mathcal{X} = TX$ be the total space of the holomorphic tangent bundle of X.
- We have
$$\begin{tikzcd} \mathcal{X} \times S \arrow[r, shift left, two heads, "\pi"] \arrow[dr, two heads, "\tilde{\pi}"'] & X \times S \arrow[l, shift left, "i"] \arrow[d, two heads, "p"] \\ & S \end{tikzcd}$$
where $i : X \times S \hookrightarrow \mathcal{X} \times S$ is the embedding to the zero section.
- So $\mathrm{R}p_* = \mathrm{R}p_* \circ \mathrm{R}\pi_* \circ \mathrm{R}i_* = \mathrm{R}\tilde{\pi}_* \circ \mathrm{R}i_*$ on the derived categories.
- We can forget $\mathrm{R}p_*$ and focus on $\mathrm{R}i_*$ and $\mathrm{R}\tilde{\pi}_*$.

Ri_*

- Let $(E^\bullet, A'') \in \mathrm{B}(X \times S)$ be a coshesive module on $X \times S$.
- We can show that $\mathrm{R}i_*(E^\bullet, A'') = (\underline{i_*}(E^\bullet), \underline{i_*}(A''))$ where
  - $\underline{i_*}(E^\bullet) = \pi^*(\wedge^\bullet TX \otimes E^\bullet)$;
  - $\underline{i_*}(A'') = \pi^* A'' + i_y$ the Koszul resolution.
- We can compute $\mathrm{R}\tilde{\pi}_*(\underline{i_*}(E^\bullet), \underline{i_*}(A''))$ as before.

Hypoelliptic superconnection

- We can use the much larger space $\mathcal{X}$.
- Let $T\mathcal{X}$ be the vertical holomorphic tangent bundle of $\mathcal{X}$.
- Let y be the tautological section of $T\mathcal{X}$ and $Y = y + \bar{y}$.
- Let (roughly)

  $$\mathcal{A}''_Y = A'' + \bar{\partial}^V + i_y$$

  and

  $$\mathcal{A}'_Y = A' + (\bar{\partial}^V)^* - \sqrt{-1}q^*\partial^X\omega_X + i_{\bar{y}} + \bar{y}_*\wedge$$
- $\mathcal{A}''_Y$ and $\mathcal{A}'_Y$ are conjugate if we replace the metric h^E by the generalized metric $h^E e^{-i\omega_X}$.

Heat kernel of the hypoelliptic superconnection

- We obtain a superconnection $\mathcal{A} = \mathcal{A}' + \mathcal{A}''$ on $\mathrm{R}\tilde{\pi}_*(\underline{i_*}(E^\bullet), \underline{i_*}(\mathcal{A}''))$, whose $\mathcal{A}^2$ is a hypoelliptic operator in the $\mathcal{X}$ direction.
- $\mathcal{A}^2 = \frac{1}{2}(-\Delta^V + |Y|^2) + \nabla_{Y^H} + \dots$
- $\exp(-t\mathcal{A}^2)$ has a smooth heat kernel, moreover, it is of trace class.
- We can define the Chern character $\mathrm{ch}(\mathcal{A}'', h^E, g^X, \omega_X)$ as before.

Theorem (Bismut-Shen-W, [Bismut et al., 2023])

- *ch*$(\mathcal{A}'', h^E, g^X, \omega_X) \in \Omega^{(=)}(S, \mathbb{R})$ *and is d-closed;*
- *ch*$(\mathcal{A}'', h^E, g^X, \omega_X) \in H^{(=)}_{BC}(S, \mathbb{R})$ *is independent of the choice of* h^E *and* g^X *and* ω_X*.*
- $$ch(\mathcal{A}'', h^E, g^X, \omega_X) = ch(Rp_*(E^\bullet, \mathcal{A}'')) \in H^{(=)}_{BC}(S, \mathbb{R})$$

Exotic hypoelliptic superconnection

- $\mathcal{A}^2$ still contains $\bar{\partial}\partial\omega_X$;
- We replace ω_X by $\tilde{\omega}_X = |Y|^2\omega_X$ and define $\mathcal{A}$ in the similar way as before. We call it the **exotic** hypoelliptic superconnection.
- The theorem in the previous slide still holds.
- The $\bar{\partial}\partial\omega_X$ terms in $\mathcal{A}^2$ are canceled.

Theorem (Bismut-Shen-W, [Bismut et al., 2023])

$$\lim_{t\to 0+} ch(\mathcal{A}'', h^E, g^X/t^3, \tilde{\omega}_X/t) = p_*[q^* Td(TX, g^X) ch(E^\bullet, A'', h^E)] \text{ as forms.}$$

- We proved the Grothendieck-Riemann-Roch theorem in full generality.

Future works?

- Explicit constructions of antiholomorphic flat superconnections;
- Noncompact case;
- Matrix factorization;
- Arakelov geometry;
- Quillen line bundle, eta invatiants;
- Other higher structures.

Thank you!

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